Reassessment of the Classical Turbulence Closures

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CHAPTER 1

INTRODUCTION

1.1 Background: the turbulence problem

The statistical theory of turbulence was formulated in 1935 by G. I. Taylor [67]. Breaking with previous heuristic mixing-length models of turbulent shear flows, Taylor introduced the more fundamental viewpoint that the velocity in a turbulent flow is a random field, each realization of which is governed by the Navier-Stokes equations. Taylor also formulated the basic problem of this theory: computing the velocity correlation function in a turbulent flow in which the mean velocity vanishes and the velocity fluctuation statistics are homogeneous and isotropic. The absence of any obvious single length or time scale rules out phenomenological mixing-length descriptions of this problem and focuses attention instead on the most important feature of turbulent flows: the existence of a continuum of length and time scales.

Unfortunately, although the kinematic restrictions of homogeneity and isotropy result in many elegant simplifications, any attempt to use the Navier-Stokes equations as the exclusive basis of the theory fails because the governing equations only provide a relation, the famous *Kármán-Howarth equation* [21], between the velocity correlation function and a third-order moment of the velocity field. Because this equation contains two unknowns, it is indeterminate. The same problem recurs in the evolution equation for moments of any order [2]. The difficulty of the statistical theory of turbulence therefore could not be more basic: its governing equations are effectively unknown.

This problem came to be called the turbulence 'closure' problem. The first attempts to solve it, by Obukhov [52], Kovasznay [35], Heisenberg [30, 29], and many others [51], obtained a determinate theory by postulating a *closure hypothesis*, a relation between the second- and third-order moments of the velocity field. These theories are the *classical turbulence closures* of the title. It must be stressed that the crucial additional relation really was *postulated*: beyond some miscellaneous heuristic considerations and obvious dimensional constraints, there was no systematic procedure leading to a unique result. These theories are given considerable attention in the older texts [51, 31]. But the accounts are discouraging; they conclude that these theories are both arbitrary and numerous, that they have little if any connection to the equations of motion, and that close scrutiny reveals that they are all unsatisfactory for one reason or another. Confirming this negative impression, some newer texts [57, 3] disregard the classical closures entirely.

Nevertheless, we will argue against neglecting the classical closures for three principal reasons. First, continued interest in the classical closure approach to turbulence theory is attested by the application of the Leith model (Section 5.3) to transient turbulence evolution [20], the generalization of the Heisenberg model (Section 5.2) to turbulence generated by linear instabilities [10], and the use of classical closures by Clark and Zemach [16, 17] and Clark [15] to evaluate simpler turbulence models. Other applications include the derivation of *multiple-scale* turbulence models from the classical Kovasznay (Section 5.1) and Leith models by Schiestel [66] and Cadiou and Hanjalic [7].

Second, the classical closures make important and useful hypotheses about energy transfer in turbulent flows. Thus, the Kovasznay model is an analytical formulation of the idea of stepwise cascade that is so often invoked in arm-waving discussions of turbulence dynamics. The Heisenberg model is an eddy viscosity model in which damping at any scale of motion is due to interaction with smaller scales of motion; it also expresses a familiar heuristic analytically. The Leith model describes energy transfer as the diffusive spreading of the excitation at any scale of motion by random straining due to larger scales of motion. At the very least, these physical pictures and their implications should be useful to students.

Third, classical closure theory can have an important role in applications. While no one denies that closure theories offer a much more realistic description of turbulence dynamics than the models typically used for engineering computations, their increased complexity and computation requirements have not seemed justifiable to Computational Fluid Dynamics (CFD) practitioners. (However, recent technological developments [62, 8] raise problems of transient turbulence evolution and of turbulence with weak, unsteady production mechanisms, which may change this evaluation.) Computation of these flows, which are often generically but imprecisely called 'non-equilibrium' turbulent flows, poses special difficulties for existing turbulence treatments. The feasibility of Direct Numerical Simulation (DNS) of turbulent flows in complex geometries remains hypothetical; it is even less likely to be practical in non-stationary flows, where computing statistics could require ensemble averaging. The simplest practical alternative, applying existing Reynolds-averaged Navier-Stokes (RANS) models, proves problematic because their very strong assumptions about fluctuation statistics, usually based on Kolmogorov's theory of the small scales in turbulence, do not apply to this class of flows. The intermediate strategy, large-eddy simulation (LES), faces problems similar to those of both DNS and RANS: because it resolves fluctuations, LES must evaluate unsteady statistics, and because it uses subgrid models, it also makes assumptions about subgrid flow statistics, again often based on Kolmogorov or other theories of self-similarity. Although both problems are certainly less severe for LES, LES calculations remain very expensive relative to RANS calculations; this fact should discourage the automatic conclusion that where RANS models fail, no fully statistical approach can succeed, and that LES is the only serious alternative.

We contend that closure theories offer a solution to these problems. Because they are *statistical* theories, temporal non-stationarity does not pose any special difficulty for them. The Kolmogorov theory and its role in turbulence modeling is a fundamental issue that requires more discussion. Certainly, Kolmogorov's ideas remain the foundation of all thinking about turbulence [26]. But it is crucial that the Kolmogorov theory describes an *attractor* for fluctuation statistics. Models typically assume much more. For example, the Smagorinsky subgrid model for LES assumes that the unresolved scales of motion are in a local Kolmogorov steady state in instantaneous energy flux balance with the resolved scales of motion, or in other words, that the Kolmogorov theory describes a permanent feature of the small scales of motion. The standard derivations of RANS models may not invoke Kolmogorov phenomenology explicitly, but by using the dissipation rate, a small-scale quantity, to determine a length or a time scale characteristic of the largest scales of motion, RANS models assume the validity at all times of Kolmogorov's central premise that the dissipation rate is scale-independent. In this case, the Kolmogorov theory is taken to be a permanent feature of all fluctuating scales of motion.

The contrast with closure theories could not be greater: *closure theories are constructed to be compatible with Kolmogorov scaling, but do not impose it in advance*. This more realistic role of the Kolmogorov theory is critical, because when turbulence evolves toward a self-similar state, or from one self-similar state to another, it must not be assumed that the intervening dynamics can be described by Kolmogorov's picture. Evidence of departure from 'equilibrium' states in the transient evolution of forced turbulence is given by Connaughton and Nazarenko [20] and in the dynamics of turbulence with time-dependent forcing by Rubinstein et al. [63]. But these considerations are very far from new; more than 30 years ago, Schiestel [66] had already stressed that modeling transient turbulence requires going well beyond the standard modeling heuristics. The applicability of Kolmogorov phenomenology even to slowly evolving flows like decaying turbulence has recently been challenged by George and Wang [28].

The feasibility of applying closure theories to non-equilibrium turbulence in engineering flows has been demonstrated by Bertoglio et al. [4], where a closure-based theory is applied to demonstrate mismatch of dissipation and energy transfer rate in steady airfoil flows. (Other computational applications of closure theories include the application of the LWN (Local Wave-Number) model, a variant of the Leith closure, to a wide variety of self-similar homogeneous sheared and strained flows by Clark [14, 15] as well as the work of Canuto and Dubovikov [9] and their collaborators on complex turbulent flows subject to shear, stratification, and rotation.)

To summarize, there is ample evidence that the additional complexity of closure models is justified in a broad range of problems in which existing models are unsatisfactory and direct simulation is impractical, and that closures can be successfully incorporated in practical CFD codes and applied to engineering problems. These considerations motivate the present 'reassessment of the classical turbulence closures.' Our goal is not only to describe and evaluate these theories, but more importantly, to update them and to urge the turbulence community to give them another chance.

1.2 Outline of the chapters

Any discussion of the classical turbulence closures must begin by addressing those limitations which raised considerable doubt about their fundamental validity. If the classical closures are defective, we should ask why they are defective and how, if possible, they can be improved.

Although motivated by an effort to understand turbulence at more fundamental level than mixing-length models, the classical approach remained entirely heuristic because the closure relations between the second- and third-order statistics were postulated without justification. Later work, closer in spirit to Taylor's original theory, sought instead to invoke an explicit statistical hypothesis to close the infinite hierarchy of moment equations generated by the Navier-Stokes equations. We will follow Orszag [55] and refer to these theories as *analytical turbulence closures.* Although the first analytical closure, the *Quasinormality* Theory, proved to be seriously defective [53, 54], subsequent theoretical developments based on the Direct Interaction Approximation (DIA) of Kraichnan [36] led to the formulation of theoretically satisfactory analytical closures including the Lagrangian History DIA (LHDIA, [37]), Lagrangian Renormalized Approximation (LRA, [33, 34]), Test-Field Model (TFM, [39]), and the Eddy-Damped Quasinormal Markovianized closure (EDQNM, [54]). Although we have cited several different analytical closures, it will be shown that they share a common underlying structure and that they are all derived in a similar way. These analytical theories overcome the objections to the classical closures, but at the expense of computational requirements that are generally viewed as excessive for practical CFD.

Our approach to updating the classical closures will be to compare them to the analytical closures. Confronting the oversimplifications of the classical closures with these more rational and comprehensive theories provides a systematic way to understand both where the classical closures fail and how they can be improved. The goal of this strategy is to develop models that combine the analytical simplicity of the classical closures with the physical realism of the analytical closures. Of course, compromises are necessary and this goal can only be imperfectly realized.

Three representative classical theories will be considered in detail: the Kovasznay, Leith, and Heisenberg closures, which are respectively the simplest algebraic, differential, and integral closure theories. Comparison with analytical closures shows first that the classical closures all give an incorrect account of the turbulent time scale. The classical closures are formulated without explicit consideration of the time scale, which they implicitly express in terms of the energy spectrum. In analytical closures, the time scale and the energy spectrum satisfy independent evolution equations. Moreover, analytical closures distinguish a modal frequency from the relaxation time for nonlinear interactions; this distinction is entirely absent in the classical closures. Our first recommendation for improving the classical closures retains their energy transfer models, but introduces time-scale models based on the structure of analytical closures. Significant improvements result from this simple change; in particular, the objection that classical closures make unphysical predictions in the dissipation range can be largely overcome.

Next, we compare the energy transfer modeling of the classical and analytical closures more carefully. Kraichnan [41] observes that all analytical theories model energy transfer *in part* as the damping of the excitation at any scale of motion by nonlinear interaction with smaller scales of motion. Analytical closures thereby justify the ideas of 'eddy damping' and 'eddy viscosity' found in the earliest mixing-length models of turbulence. Following Kraichnan [42], we show that the Heisenberg model describes a limiting form of this type of nonlinear interaction. But this comparison also discloses a crucial shortcoming of the Heisenberg model: analytical closures also include a second kind of energy transfer that corresponds to what is now often called, following Leith, energy 'backscatter.' Whereas the Heisenberg model forces all energy transfer to be from large to small scales, backscatter allows 'reverse' transfer from small to large scales. This transfer mechanism is both physically and analytically distinct from eddy damping; it is neither a 'negative eddy viscosity' nor even a diminished eddy viscosity. Adding a backscatter term to the Heisenberg model results in the model of Rubinstein and Clark [60]. In its treatment of energy backscatter, this model can be compared to the closure of Canuto and Dubovikov [9]. The substantial improvements due to this simple modification of the Heisenberg model are described in detail. It should be stressed that the analytical form of backscatter transfer cannot be guessed by elementary arguments; a theory is necessary.

This analysis also reveals an important theoretical limitation of the Kovasznay model and the associated picture of stepwise cascade. The Kovasznay model can be formulated as a limit of the Heisenberg model in which increasingly short-range interactions are dominant. The structure of the Kovasznay model precludes any distinction between eddy damping and backscatter. Thus, although the picture of stepwise cascade is simple, attractive, and perhaps somewhat useful heuristically, it must not be taken too literally. The same limit of the model of Rubinstein and Clark [60] does not lead to the Kovasznay model, but to a modified Leith model.

A very similar modification of the Leith model had already been proposed on heuristic grounds and tested extensively by Clark [15]. The Leith model has also been connected to analytical closures by Weinstock [68] through different approximation procedures.

We conclude that by updating the classical closures with suitable elements from analytical closure, we obtain greatly improved models which are analytically no more complicated than the original classical models. These new models share a one-dimensional picture of energy transfer with the classical models, a drastic simplification which makes them easy to compute, yet they include the basic processes of temporal decorrelation and energy transfer suggested by the considerably more complex analytical closures. We believe that this approach extracts useful practical information from analytical closures, which have hitherto been restricted largely to theoretical studies.

As preliminary validation studies, we test the models in the two simplest problems of isotropic turbulence: decaying isotropic turbulence, and steady-state isotropic turbulence maintained by forcing at large scales. At this point, we are attempting to demonstrate that the spectral dynamics are qualitatively correct, not to validate any particular model; therefore, comparisons with DNS or more elaborate closures will not be given.

The numerical studies demonstrate that despite their known shortcomings, the classical closures can successfully predict certain important general properties of these turbulent flows, including the onset of power-law decay and the development of a Kolmogorov steady state in forced turbulence. However, the absence of of energy backscatter in the unmodified classical models and in the modified Kovasznay and Heisenberg models produces unrealistic behavior in the large scales in forcing simulations. Although the Leith model is somewhat better in this respect, only the model of Rubinstein and Clark [60] gives entirely satisfactory predictions for the large scales of motion. This model also reproduces refined features like the 'bottleneck' phenomenon: the tendency of inertial range energy to pile up immediately before the dissipation range [24]. This phenomenon cannot be predicted with purely local energy transfer models based on the Kovasznay and Leith closures.

An outline of the contents follows. Chapter 2 reviews the statistical formulation of the turbulence problem, including the Fourier transform formalism and the basic (unclosed) governing equation, the Lin Equation. Chapter 3 reviews the analytical properties and basic constraints that we will impose: the existence of a Kolmogorov steady-state constant flux solution, reasonable predictions in the dissipation range, and the existence of equipartition solutions. Some of the elementary classical models are discussed next: frequency-based models such as the Kovasznay model in Chapter 4, Leith diffusion models in Chapter 5, and eddy-viscosity models, including the Heisenberg model, in Chapter 6. Chapter 7 is a summary of the analytical theories that we will use to enhance the elementary models; these include the Quasinormality Theory, the EDQNM, and finally the DIA. Chapter 8 discusses how these more complete models can be used to derive new eddy-viscosity models; similarly, Chapter 9 shows how Leith diffusion models can be understood in terms of analytical closures. Time-scale modeling, an important ingredient in analytical closures that is missing in the classical closures, is discussed in Chapter 10. Chapter 11 presents numerical solutions of various models, illustrating production-equals-dissipation equilibria, free-decay behavior, and equipartition properties.

1.3 Notation

In some general derivations, vectors will be written with bold-face italics; thus, the velocity vector is denoted by $\mathbf{u}$. Similarly, second-rank tensors will be written in a sans-serif font, so that the velocity correlation is denoted by U. This notation emphasizes that these are physically meaningful quantities apart from their scalar components. However, in some cases, this notation becomes ambiguous. If so, component notation is used, so that the velocity is denoted by u_i and the correlation tensor by U_{ij}. In some arguments, it will be convenient to denote any unknown constant by C, with the understanding that C does not necessarily denote the same constant each time it appears. We hope that these notational conventions will not be confusing.

CHAPTER 2

STATISTICAL FORMULATION

This chapter presents a brief outline of the steps leading from the Navier-Stokes equations to the governing statistical equation for the turbulent energy spectrum; the goal is to arrive at the results as expeditiously as possible and to 'fix notation' and terminology. Details and more careful treatment of mathematical technicalities are available in references like Batchelor [2] and Orszag [55].

Recall that Taylor's statistical theory called for the computation of the velocity correlation function in homogeneous, isotropic turbulence in free (unbounded) three-dimensional Euclidean space. The velocity field satisfies the incompressible Navier-Stokes equations:

$$\dot{\boldsymbol{u}} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2.2}$$

We recall that the 'pressure' term p can be eliminated from Eq. (2.1) by taking the divergence and using Eq. (2.2). Then replacing p by this result gives an equation in terms of velocity $\boldsymbol{u}$ alone,

$$\dot{u}_i + u_p \frac{\partial u_i}{\partial x_p} = \frac{\partial}{\partial x_i} \nabla^{-2} \left(\frac{\partial u_m}{\partial x_n} \frac{\partial u_n}{\partial x_m} \right) + \nu \nabla^2 u_i \tag{2.3}$$

a form which exhibits the spatial nonlocality of the incompressible Navier-Stokes equations.

The (single-time) velocity correlation in general is a tensor function of two arguments:

$$U_{ij}(\boldsymbol{x}, \boldsymbol{x}'; t) = \langle u_i(\boldsymbol{x}, t) u_j(\boldsymbol{x}', t) \rangle \tag{2.4}$$

Here we have indicated the time argument to emphasize that the velocities are taken at the same time; however, henceforth the time argument will be assumed and not written explicitly. Spatial homogeneity means that $U_{ij}(\boldsymbol{x}, \boldsymbol{x}')$ depends only on the difference $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$:

$$U_{ij}(\boldsymbol{x}, \boldsymbol{x}') = U_{ij}(\boldsymbol{x} - \boldsymbol{x}') = U_{ij}(\boldsymbol{r}) \tag{2.5}$$

or equivalently that

$$U_{ij}(\boldsymbol{r}) = \langle u_i(\boldsymbol{x}) u_j(\boldsymbol{x} + \boldsymbol{r}) \rangle \tag{2.6}$$

does not depend on $\boldsymbol{x}$.

Following Batchelor [2], the evolution equation for $U_{ij}(\boldsymbol{r})$ is obtained by multiplying Eq. (2.1) for $\boldsymbol{u}(\boldsymbol{x})$ by $\boldsymbol{u}(\boldsymbol{x}')$, multiplying Eq. (2.1) for $\boldsymbol{u}(\boldsymbol{x}')$ by $\boldsymbol{u}(\boldsymbol{x})$, and adding the results. The conclusion has the form

$$\dot{U}_{ij}(\boldsymbol{r}) = T_{ij}(\boldsymbol{r}) + \Pi_{ij}(\boldsymbol{r}) + 2\nu \nabla^2 U_{ij}(\boldsymbol{r}) \tag{2.7}$$

(note that $\nabla^2 = \partial/\partial r_p \partial/\partial r_p$) where

$$\begin{aligned} T_{ij}(\boldsymbol{r}) = & \left\langle u_p(\boldsymbol{x}) \frac{\partial u_i(\boldsymbol{x})}{\partial x_p} u_j(\boldsymbol{x}') \right\rangle \\ & + \left\langle u_p(\boldsymbol{x}') \frac{\partial u_j(\boldsymbol{x}')}{\partial x'_p} u_i(\boldsymbol{x}) \right\rangle \end{aligned} \tag{2.8}$$

$$\Pi_{ij}(\boldsymbol{r}) = \left\langle u_j(\boldsymbol{x}') \frac{\partial p(\boldsymbol{x})}{\partial x_i} \right\rangle + \left\langle u_i(\boldsymbol{x}) \frac{\partial p(\boldsymbol{x}')}{\partial x'_j} \right\rangle \tag{2.9}$$

The expressions for $T_{ij}(\boldsymbol{r})$ and $\Pi_{ij}(\boldsymbol{r})$ in terms of $\boldsymbol{x}$ and $\boldsymbol{x}'$ prove more convenient for the following derivations. Let's focus on the *inviscid* ($\nu = 0$) form of Eq. (2.7). Following Batchelor [2], remark first that the trace of $\Pi_{ij}(\boldsymbol{r})$, $\delta_{ij}\Pi_{ij}(\boldsymbol{r})$, vanishes for all $\boldsymbol{r}$. To prove it, setting $i = j$ in the first term on the right side of Eq. (2.9),

$$\left\langle u_i(\boldsymbol{x}') \frac{\partial p(\boldsymbol{x})}{\partial x_i} \right\rangle = \frac{\partial}{\partial x_i} \langle u_i(\boldsymbol{x}') p(\boldsymbol{x}) \rangle \tag{2.10}$$

But since writing

$$\langle u_i(\boldsymbol{x}') p(\boldsymbol{x}) \rangle = \Upsilon_i(\boldsymbol{x}' - \boldsymbol{x}) \tag{2.11}$$

it follows that

$$\begin{aligned}\left\langle u_i(\boldsymbol{x}')\frac{\partial p(\boldsymbol{x})}{\partial x_i}\right\rangle &= \frac{\partial}{\partial x_i}\Upsilon_i(\boldsymbol{x}'-\boldsymbol{x}) = -\frac{\partial}{\partial x_i'}\Upsilon_i(\boldsymbol{x}'-\boldsymbol{x}) \\ &= -\frac{\partial}{\partial x_i'}\left\langle u_i(\boldsymbol{x}')p(\boldsymbol{x})\right\rangle \\ &= -\left\langle \frac{\partial u_i(\boldsymbol{x}')}{\partial x_i'}p(\boldsymbol{x})\right\rangle = 0\end{aligned} \tag{2.12}$$

The same argument proves that

$$\left\langle u_i(\boldsymbol{x})\frac{\partial p(\boldsymbol{x}')}{\partial x_i}\right\rangle = 0 \tag{2.13}$$

hence

$$\delta_{ij}\Pi_{ij}(\boldsymbol{r}) = 0 \tag{2.14}$$

for all $\boldsymbol{r}$, as asserted.

Taking the trace of Eq. (2.7) will eliminate the pressure term; denoting the trace of $\mathsf{U}(\boldsymbol{r})$ and $\mathsf{T}(\boldsymbol{r})$ by $U(\boldsymbol{r})$ and $T(\boldsymbol{r})$ respectively,

$$\dot{U}(\boldsymbol{r}) = T(\boldsymbol{r}) \tag{2.15}$$

Next we note that

$$T_{ij}(\boldsymbol{r})|_{\boldsymbol{r}=0} = 0 \tag{2.16}$$

for setting $\boldsymbol{r} = 0$ in the definition Eq. (2.8) just means equating $\boldsymbol{x}' = \boldsymbol{x}$. Then

$$\begin{aligned}T_{ij}(\boldsymbol{r})|_{\boldsymbol{r}=0} &= \left\langle u_p(\boldsymbol{x})(\partial u_i(\boldsymbol{x})/\partial x_p)u_j(\boldsymbol{x})\right\rangle \\ &+ \left\langle u_p(\boldsymbol{x})(\partial u_j(\boldsymbol{x})/\partial x_p)u_i(\boldsymbol{x})\right\rangle \\ &= (\partial/\partial x_p)\left\langle u_p(\boldsymbol{x})u_i(\boldsymbol{x})u_j(\boldsymbol{x})\right\rangle = 0\end{aligned} \tag{2.17}$$

due to homogeneity. Then in particular,

$$\dot{U}(\boldsymbol{r})|_{\boldsymbol{r}=0} = 0 \tag{2.18}$$

Since

$$U(\boldsymbol{r})|_{\boldsymbol{r}=0} = \left\langle u_p(\boldsymbol{x})u_p(\boldsymbol{x})\right\rangle \tag{2.19}$$

Eq. (2.18) states the conservation of energy by the nonlinear interactions. Assuming the smoothness of statistics, it is possible to write

$$T(\boldsymbol{r}) = \boldsymbol{r} \cdot \boldsymbol{F}(\boldsymbol{r}) \tag{2.20}$$

whence, restoring the viscosity term,

$$\dot{U}(\boldsymbol{r}) = \boldsymbol{r} \cdot \boldsymbol{F}(\boldsymbol{r}) + 2\nu\nabla^2 U(\boldsymbol{r}) \tag{2.21}$$

At this point, this will be all that is required; the explicit forms of neither $T(\boldsymbol{r})$ nor $F(\boldsymbol{r})$ will have any role for us.

We next introduce the Fourier formulation. Our treatment will be heuristic, ignoring certain mathematical refinements, which are carefully treated in Orszag's lectures [55]. In this formalism, the velocity is decomposed into Fourier modes as

$$\boldsymbol{u}(\boldsymbol{x}) = \int d\boldsymbol{k}\; \boldsymbol{u}(\boldsymbol{k}) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \tag{2.22}$$

This step can be motivated on physical grounds as a way to describe the velocity field in terms of 'scales of motion' in which the mode $\exp(i\boldsymbol{k} \cdot \boldsymbol{x})$ self-evidently has spatial scale $2\pi/|\boldsymbol{k}|$. This property of Fourier modes then permits the discussion of turbulence in terms of energy transfer among modes. It is probably because of this fact that all classical closures were formulated in terms of Fourier variables.

Further mathematical motivation derives from a basic fact about the Navier-Stokes equations in unbounded space, namely translation invariance: if we know that $\boldsymbol{u}(\boldsymbol{x}, t)$ satisfies the equations, then so does $\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{a}, t)$ for any fixed $\boldsymbol{a}$. (Note again the importance of the assumption of free unbounded space). The effect of translation is a phase change of the Fourier amplitude, *viz.*,

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{a}) &= \int d\boldsymbol{k}\; \boldsymbol{u}(\boldsymbol{k}) \exp(i\boldsymbol{k} \cdot (\boldsymbol{x} + \boldsymbol{a})) \\ &= \int d\boldsymbol{k}\; \big(\exp(i\boldsymbol{k} \cdot \boldsymbol{a})\boldsymbol{u}(\boldsymbol{k})\big) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \end{aligned} \tag{2.23}$$

This 'diagonalization' of the operation of translation (that is, its expression in the Fourier basis as multiplication by a scalar function) extends to differentiation by

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{u}(\boldsymbol{x}) &= \int d\boldsymbol{k}\; \boldsymbol{u}(\boldsymbol{k}) \frac{\partial}{\partial \boldsymbol{x}} \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \\ &= \int d\boldsymbol{k}\; \big(i\boldsymbol{k}\boldsymbol{u}(\boldsymbol{k})\big) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \end{aligned} \tag{2.24}$$

Eqs. (2.23) and (2.24) are of course related because differentiation is basically an infinitesimal translation. The simple expression for the divergence follows:

$$\nabla \cdot \boldsymbol{u}(\boldsymbol{x}) = \int d\boldsymbol{k}\; i\big(\boldsymbol{k} \cdot \boldsymbol{u}(\boldsymbol{k}, t)\big) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \tag{2.25}$$

This formula calls attention to an important detail: the Fourier amplitudes $\boldsymbol{u}(\boldsymbol{k})$ are complex. Since the original velocity field $\boldsymbol{u}(\boldsymbol{x})$ is real, Eq. (2.22) shows that

$$\boldsymbol{u}(-\boldsymbol{k}) = \boldsymbol{u}(\boldsymbol{k})^* \tag{2.26}$$

Eq. (2.26) implies that Eq. (2.22) is basically a 'cosine' transformation; the factor i in Eq. (2.24) arises because the derivative is a 'sine' transformation. However, it is unnecessary to insist on these minutiae.

An important property of Fourier amplitudes follows from spatial homogeneity, namely since

$$\begin{aligned} &\boldsymbol{u}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{a}) = \\ &\quad \int d\boldsymbol{k}\; \boldsymbol{u}(\boldsymbol{k}) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \int d\boldsymbol{k}'\; \boldsymbol{u}(\boldsymbol{k}') \exp(i\boldsymbol{k}' \cdot (\boldsymbol{x}+\boldsymbol{a})) \end{aligned} \tag{2.27}$$

then

$$\begin{aligned} &\langle \boldsymbol{u}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{a})\rangle = \\ &\quad \int d\boldsymbol{k}d\boldsymbol{k}' \langle \boldsymbol{u}(\boldsymbol{k})\boldsymbol{u}(\boldsymbol{k}')\rangle \exp(i\boldsymbol{x} \cdot (\boldsymbol{k}+\boldsymbol{k}')) \exp(i\boldsymbol{k}' \cdot \boldsymbol{a})) \end{aligned} \tag{2.28}$$

Since the left side is independent of $\boldsymbol{x}$, then necessarily $\boldsymbol{k} + \boldsymbol{k}' = 0$. This is another 'diagonalization' property, namely that only modes with opposite wavevectors are correlated with each other; equivalently, in view of Eq. (2.26), each Fourier mode can only be correlated with its complex conjugate. Another restatement is that the Fourier modes define the *proper orthogonal decomposition* of a statistically homogeneous velocity field. A similar argument demonstrates that any correlation $\langle \boldsymbol{u}(\boldsymbol{k}_1) \cdots \boldsymbol{u}(\boldsymbol{k}_n)\rangle$ vanishes unless $\boldsymbol{k}_1 + \cdots \boldsymbol{k}_n = 0$.

Eq. (2.28) requires that

$$\langle \boldsymbol{u}(\boldsymbol{k})\boldsymbol{u}(\boldsymbol{k}')\rangle = \mathsf{U}(\boldsymbol{k})\delta(\boldsymbol{k} - \boldsymbol{k}') \tag{2.29}$$

because substituting this result in Eq. (2.28) gives the modal decomposition of $\mathsf{U}(\boldsymbol{r})$:

$$\begin{aligned} \mathsf{U}(\boldsymbol{r}) &= \langle \boldsymbol{u}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r})\rangle \\ &= \int d\boldsymbol{k}\; \mathsf{U}(\boldsymbol{k}) \exp(i\boldsymbol{k} \cdot \boldsymbol{r}) \end{aligned} \tag{2.30}$$

We will refer to $U_{ij}(\boldsymbol{k})$ as the *modal energy density*. Eq. (2.29) can be demonstrated using inverse transforms [55]). In particular, taking the trace of both sides,

$$U(\boldsymbol{r}) = \int d\boldsymbol{k}\; U(\boldsymbol{k}) \exp(i\boldsymbol{k} \cdot \boldsymbol{r}) \tag{2.31}$$

This result lets us express the mean kinetic energy of the velocity fluctuations in terms of the trace of $\mathsf{U}(\boldsymbol{k})$ by

$$\mathsf{k} = \frac{1}{2} \int d\boldsymbol{k}\; U(\boldsymbol{k}) \tag{2.32}$$

We can now exhibit the decomposition of Eq. (2.21) into Fourier modes. Let

$$\mathsf{F}(\boldsymbol{r}) = \int d\boldsymbol{k}\; \mathsf{F}(\boldsymbol{k}) \exp(i\boldsymbol{k} \cdot \boldsymbol{r}) \tag{2.33}$$

Then

$$\begin{aligned} \boldsymbol{r} \cdot \mathsf{F}(\boldsymbol{r}) &= \boldsymbol{r} \cdot \int d\boldsymbol{k}\; \mathsf{F}(\boldsymbol{k}) \exp(i\boldsymbol{k} \cdot \boldsymbol{r}) \\ &= \int d\boldsymbol{k}\; \mathsf{F}(\boldsymbol{k}) \cdot (1/i) \frac{\partial}{\partial \boldsymbol{k}} \exp(i\boldsymbol{k} \cdot \boldsymbol{r}) \\ &= \int d\boldsymbol{k}\; \left(i \frac{\partial}{\partial \boldsymbol{k}} \cdot \mathsf{F}(\boldsymbol{k}) \right) \exp(i\boldsymbol{k} \cdot \boldsymbol{r}) \end{aligned} \tag{2.34}$$

Equating coefficients of like Fourier modes,

$$\dot{U}(\boldsymbol{k}) = i \frac{\partial}{\partial \boldsymbol{k}} \cdot \mathsf{F}(\boldsymbol{k}) - 2\nu k^2 U(\boldsymbol{k}) \tag{2.35}$$

At this point, we invoke the restriction to isotropy. Isotropy requires first that

$$\mathsf{F}(\boldsymbol{k}) = \boldsymbol{k} F(\boldsymbol{k}) \tag{2.36}$$

and second that both $U(\boldsymbol{k})$ and $F(\boldsymbol{k})$ depend only on $k = |\boldsymbol{k}|$. Then

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{k}} \cdot \mathsf{F}(\boldsymbol{k}) &= \frac{\partial}{\partial \boldsymbol{k}} \cdot \big(\boldsymbol{k} F(k)\big) \\ &= 3F(k) + \boldsymbol{k} \cdot F'(k)(\boldsymbol{k}/k) = 3F(k) + kF'(k) \end{aligned} \tag{2.37}$$

Combining all of these results,

$$\dot{U}(k) = i\big(3F(k) + kF'(k)\big) - 2\nu k^2 U(k) \tag{2.38}$$

Applying Eq. (2.32) in the isotropic case,

$$\begin{aligned} \mathsf{k} = \frac{1}{2}\int d\boldsymbol{k}\; U(\boldsymbol{k}) &= \frac{1}{2}\int dk \oint d\Omega\, U(k) \\ &= 2\pi \int k^2 U(k) dk \end{aligned} \tag{2.39}$$

Expressing Eq. (2.32) in terms of the energy spectrum,

$$E(k) = 2\pi k^2 U(k) \tag{2.40}$$

$$\begin{aligned} \dot{E}(k) = 2\pi i\big(3k^2F(k) + k^3F'(k)\big) - 2\nu k^2 E(k) \\ = -\frac{\partial}{\partial k}\mathcal{F}(k) - 2\nu k^2 E(k) \end{aligned} \tag{2.41}$$

where

$$\mathcal{F}(k) = -2\pi i k^3 F(k) \tag{2.42}$$

The choice of sign will be motivated shortly. But we must emphasize that since $\mathcal{F}(k)$ originates with a third-order velocity correlation, its sign is indefinite.

It is customary in studies of turbulence closure theories to include a forcing term to generate fluctuations; such a forcing is basically a model of processes, like breakdown of instabilities, that generate turbulence naturally. By itself, Eq. (2.41) is incomplete because it can only describe decay from an arbitrary initial condition. Then the equation we will investigate, sometimes called the *Lin equation*, is

$$\dot{E}(k) = P(k) - \frac{\partial}{\partial k}\mathcal{F}(k) - 2\nu k^2 U(k) \tag{2.43}$$

2.1 Elementary properties of the Lin equation

Of course, as it stands, Eq. (2.43) is not a theory of turbulence, or of anything else for that matter, since it contains two unknowns $E(k)$ and $\mathcal{F}(k)$. However, some useful conclusions can be drawn from it. Integration over all k gives the energy balance

$$\dot{\mathsf{k}} = P - \epsilon \tag{2.44}$$

where

$$\mathsf{k} = \int_0^\infty dk\ E(k) \tag{2.45}$$

is the total kinetic energy of turbulence, and where P and ϵ are the total power input and energy dissipation rate respectively, defined as

$$P = \int_0^\infty P(k)dk \tag{2.46}$$

$$\epsilon = 2\nu \int_0^\infty k^2 E(k)dk \tag{2.47}$$

In concluding Eq. (2.44), we have assumed that

$$\mathcal{F}(k)\,|_{k=0} = \mathcal{F}(k)\,|_{k=\infty} = 0 \tag{2.48}$$

Without insisting on details, we refer to the expression for $F(k)$ as a velocity correlation, which can be expected to vanish at infinitely large and small scales.

In the absence of production and dissipation, the Lin equation is just the conservation law

$$\dot{E}(k) = -\frac{\partial}{\partial k}\mathcal{F}(k) \tag{2.49}$$

in which, in the usual understanding of the terms, $E(k)$ is the density of a conserved quantity (energy), and $\mathcal{F}$ is its flux. Integrating Eq. (2.49) over the interval $(0, k)$,

$$\frac{d}{dt}\int_0^k d\kappa\ E(\kappa) = -\mathcal{F}(k) \tag{2.50}$$

and integrating over (k, ∞),

$$\frac{d}{dt}\int_k^\infty d\kappa\ E(\kappa) = \mathcal{F}(k) \tag{2.51}$$

Under the sign convention adopted in Eq. (2.42), $\mathcal{F}(k) > 0$ means that energy is removed from larger scales $\kappa < k$ and transferred to smaller scales $\kappa > k$. By longstanding common agreement, this situation is called *forward* energy transfer, and the opposite condition $\mathcal{F}(k) < 0$ in which energy is transferred from smaller to larger scales is called *backward* energy transfer.

Integrating Eq. (2.43) with forcing and dissipation restored gives the partial energy equation

$$\frac{d}{dt}\int_0^k d\kappa\, E(\kappa) = \int_0^k d\kappa\, P(\kappa) - \int_0^k d\kappa\, 2\nu\kappa^2 E(\kappa) - \mathcal{F}(k) \tag{2.52}$$

and the complementary result

$$\frac{d}{dt}\int_k^\infty d\kappa\, E(\kappa) = \int_k^\infty d\kappa\, P(\kappa) - \int_k^\infty d\kappa\, 2\nu\kappa^2 E(\kappa) + \mathcal{F}(k) \tag{2.53}$$

In standard turbulence folklore, the forcing $P(k)$ is assumed to be concentrated at large scales $k \approx 0$. The definition Eq. (2.47) shows that on the other hand, ϵ is determined by small scales. Then if k is larger than the largest scale at which $P(\kappa) > 0$, Eq. (2.52) becomes

$$\frac{d}{dt}\int_0^k d\kappa\, E(\kappa) \approx P - \mathcal{F}(k) \tag{2.54}$$

and if k is outside the region of significant dissipation, Eq. (2.53) becomes

$$\frac{d}{dt}\int_k^\infty d\kappa\, E(\kappa) \approx \mathcal{F}(k) - \epsilon \tag{2.55}$$

If k can be chosen to satisfy both conditions, then in a steady state,

$$P = \mathcal{F}(k) = \epsilon \tag{2.56}$$

This result states the existence of a *constant flux* in a range of scales in which both forcing, or 'production' of turbulence, and dissipation can be neglected. But once again, no further conclusion can be drawn until some analytical expression is given for the flux $\mathcal{F}$. This will be the next goal.

CHAPTER 3

ANALYTICAL PROPERTIES OF THE MODELS

Recall that the basic equation of the statistical theory, the *Lin equation*

$$\dot{E}(k) = P(k) - \frac{\partial}{\partial k}\mathcal{F}(k) - 2\nu k^2 E(k) \tag{3.1}$$

contains two unknowns: E and $\mathcal{F}$ (recall that $P(k)$ is a known forcing spectrum). The simplest way to turn Eq. (3.1) into a determinate equation is to postulate a 'closure': a relation between E and $\mathcal{F}$. We will investigate closure hypotheses of the general form

$$\mathcal{F}(k,t) = \mathcal{F}[E(p,t);p]|_{0\le p<\infty} \tag{3.2}$$

so that the flux is some functional of the energy spectrum E; note that the symbol $\mathcal{F}$ denotes both the *functional* $\mathcal{F}[E]$ and the *function* of wavenumber $\mathcal{F}(k)$. In the absence of any widely accepted standard, the improvised notation is meant to indicate that $\mathcal{F}(k)$ can depend on any functional of the entire energy spectrum and all wavenumbers.

Let us note a few possibilities that this flux *ansatz* rules out. The time variable is written explicitly to emphasize that although the flux functional can depend on k, any time dependence of the flux is due entirely to time dependence of the energy spectrum. A similar caveat explains why, although it enters the Navier-Stokes equations, viscosity does not appear in Eq. (3.2): any dependence of the flux on viscosity arises from such dependence of the energy spectrum. Finally, the flux at time t depends only on the energy spectrum at time t; time history integrals are excluded.

Since it may be helpful to give some examples, we state the simplest algebraic, differential, and integral closure models of the type permitted by

Eq. (3.2), namely the Kovasznay, Leith, and Heisenberg models respectively:

$$\begin{aligned}
&\text{Kovasznay} \quad \mathcal{F}[E] = \mathsf{Ko}\left[k^5 E(k)\right]^{1/2} E(k) \\
&\text{Leith} \quad \mathcal{F}[E] = \mathsf{Le}\left[k^5 E(k)\right]^{1/2}\left[E(k) - \alpha k \frac{\partial}{\partial k} E(k)\right] \\
&\text{Heisenberg} \quad \mathcal{F}[E] = \mathsf{He}\int_0^k d\kappa\, \kappa^2 E(\kappa) \int_k^\infty dp\, \sqrt{p^{-3} E(p)}
\end{aligned}$$

The constants Ko, Le, and He will be fixed by requiring agreement with the Kolmogorov inertial range spectrum. Note that, as formulated, the Leith model contains two disposable constants: Le and α.

Eq. (3.2) defines what we understand by a 'classical' turbulence closure. It identifies a very restricted class of models. Thus, Kraichnan's Direct Interaction Approximation is excluded because it includes two-time dependence and time history integration, complex geometric factors depending on triads of wavevectors $(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$, and additional descriptors of turbulence besides the energy spectrum.

The next step is to consider how we should evaluate these or similar models: what properties do we require, expect, or merely desire? Are there constraints which rule out some models, or at least suggest the superiority of others? We will identify and describe the most important properties and constraints that will be investigated.

3.1 Existence of a local Kolmogorov constant flux state

The first and most important constraint is that a flux model should be consistent with the well-known steady state *Kolmogorov spectrum*, now treated as a part of what is called 'K41' (but we will avoid this terminology). Let us briefly review the pertinent ideas. If we neglect the production and dissipation spectra in Eq. (3.1), then in a steady state, the Lin equation reduces to the constant flux condition

$$\frac{\partial}{\partial k}\mathcal{F}(k) = 0 \tag{3.3}$$

This state is maintained by energy input at a constant rate P, and by removal at a constant rate ϵ, so that

$$P = \mathcal{F} = \epsilon \tag{3.4}$$

by energy conservation. A range of wavenumbers consistent with these conditions is called the *inertial range*. Kolmogorov's hypothesis is that in such an inertial range, the energy spectrum takes the universal form

$$E(k) = C_K \epsilon^{2/3} k^{-5/3} \tag{3.5}$$

with C_K a constant that is the same in all turbulent flows to which the hypotheses apply. According to Eq. (3.4), if $\mathcal{F}$ is positive, then under the sign convention adopted in section 2.1, the transfer of energy in the inertial range is from large to small scales. Note that under the stated conditions, whether or not they are deemed realistic, the existence of a constant flux state is not in doubt (an illuminating comparison to the famous 4/5 law is given by Sagaut and Cambon [65]); what is nontrivial is the asserted form of the energy spectrum.

But Kolmogorov's theory actually requires more, and this raises the issue of how exactly the constant flux state is maintained. For example, we could imagine that the steady state is maintained by an input energy flux at scale k_0 and energy removal at scale k_1; or in other words, by flux boundary conditions $\mathcal{F}(k_0) = \mathcal{F}(k_1) = \mathcal{F}$. The 'dissipation' would simply be the rate at which energy is removed. The key requirement is that the resulting spectrum does not depend on either k_0 or k_1. For example, dependence on the ratio k_1/k_0 would not be implausible and is certainly admissible on dimensional grounds, but it is ruled out in Kolmogorov's theory. Then the Kolmogorov spectrum Eq. (3.5) could even extend over all scales $0 \leq k < \infty$.

In reality, steady state turbulent fluctuations require some energy source and some damping mechanism. Then all of the terms in Eq. (3.1) should be considered. In the standard understanding, each term is believed to be dominant in a different range of scales: $P(k)$ at large scales where $P(k)$ is large, i.e. 'production range'; viscosity at small scales where viscous dissipation is large, i.e. 'dissipation range'; and the intermediate range of scales where energy transfer is dominant, i.e. the 'inertial range.' Then we have turbulence production at large scales, energy-conserving transfer from large to small scales, and dissipation of turbulence by viscosity at small scales.

These ideas cause much vexation in 'the literature' because the dissipation spectrum $2\nu k^2 E(k)$ does not vanish anywhere that $E(k)$ is nonzero, and there is always suspicion that any forcing that is driving the turbulence will exert some effect on all scales of motion, even where $P(k) = 0$ in Eq. (3.1). Furthermore, there could be effects of interactions between inertial range and dissipation range scales.

However, as in the previous discussion of flux boundary conditions, it is crucial that these effects do not invalidate Eq. (3.5). Suppose then that whatever production mechanism is present can be characterized by a scale k_0. Dissipation effects are quantified by the so-called Kolmogorov scale k_d at which they become comparable to nonlinearity:

$$k_d \sim \left(\frac{\epsilon}{\nu^3}\right)^{1/4} \tag{3.6}$$

This essentially dimensional result is obtained, for example by equating the inviscid frequency $\epsilon^{1/3}k^{2/3}$ to the inverse viscous damping time νk^2 at $k = k_d$. Then $\epsilon^{1/3}k_d^{2/3} \sim \nu k_d^2$ gives Eq. (3.6).

Now suppose that the steady-state energy spectrum depends in some unspecified way on the two scales k_0 and k_d; then Eq. (3.5) should be generalized to reflect this dependence, which is made explicit by writing the energy spectrum as $E(k, \epsilon|k_0, k_d)$. We will require our models to satisfy the limit

$$\begin{aligned} \lim_{k_0 \to 0, k_d \to \infty} E(k, \epsilon|k_0, k_d) &= E(k, \epsilon) \\ &= C_K \epsilon^{2/3} k^{-5/3} \end{aligned} \tag{3.7}$$

where the limit is understood to be taken so that ϵ is constant. This property is usually called 'locality' and is crucial to the formulation of models. We stress that we are simply imposing this Kolmogorovian locality on our models, not asserting that it is known to be valid; indeed, locality of higher order statistics is very likely not true.

It should be stressed that the limit we have proposed, Eq. (3.7) with constant dissipation rate, assumes that the dissipation rate can be constant if the viscosity approaches zero. This property is sometimes called (for reasons having nothing to do with turbulence) the *dissipative anomaly*, and pinpoints an important technicality about turbulence. A heuristic argument in its favor is that according to Eq. (3.6), as $\nu \to 0$, $k_d \to \infty$; then turbulence can maintain a constant dissipation rate by creating new small scales without changing the rest of the spectrum. Needless to say, direct verification is not simple (or perhaps even feasible) either experimentally or numerically, since it would require unlimited resolution of small scales; thus, this famous 'anomaly' remains another unproven, even if amply supported, hypothesis about turbulence.

The dissipative anomaly also helps resolve issue of the nonvanishing dissipation spectrum. It implies that the dissipation in any finite range of scales can vanish in the limit of vanishing viscosity:

$$\lim_{\nu \to 0} \int_{k_1}^{k_2} 2\nu\kappa^2 E(\kappa)\, d\kappa = 0 \tag{3.8}$$

whilst the dissipation itself remains constant.

If the effects of k_0 and k_d do indeed vanish as asserted in Eq. (3.7), an important question is, how rapidly do they vanish? This is will be a question for all of the closure models we analyze. Finally, note that we are insisting only on the consistency of closure models with Kolmogorov's theory; whether the 5/3 spectrum is an 'attractor' under constant forcing from an arbitrary initial condition is, at this point, a subject for numerical study.

3.2 The dissipation range

Of course, the inertial range is not everything; to investigate life beyond the inertial range, the terms in the Lin equation that were ignored previously should be restored. Consider then the steady-state equation including the viscous term

$$\frac{\partial}{\partial k}\mathcal{F}(k) + 2\nu k^2 E(k) = 0 \tag{3.9}$$

where the steady state is maintained by some unspecified energy source at scales much larger than those under consideration. Writing this equation invokes locality by taking the limit $k_0 \to 0$. We anticipate that the solution can transition from the K41 spectrum to more rapid decay in the dissipation range beyond a wavenumber of the order of the Kolmogorov scale k_d.

The earliest analyses of the classical closures, Monin and Yaglom [51] discussed the consistency of each model with the Kolmogorov steady state and then analyzed the plausibility of the model when viscosity is restored. The reason was perhaps not so much the importance of the dissipation range spectrum, as that the dissipation range dynamics as described by Eq. (3.9) are not amenable to the straightforward dimensional arguments that prove sufficient for the inertial range. Thus, plausibility of the dissipation range prediction is a more stringent test of a model than consistency with Kolmogorov scaling alone. Ideally, we would like to solve Eq. (3.9) for all k and demonstrate recovery of the inertial range when $\nu \to 0$, but we will at least try to exhibit the solution in the range $k > k_d$.

We have stressed the *plausibility* of the dissipation range predictions rather than any particular target solution. This perhaps sets the bar rather low, so for completeness we note that it is believed that at large k, $E(k)$ should decay according to an exponential law of the form

$$E(k) \sim \mathsf{C}k^{\alpha}\exp(-\mathsf{C}'(k/k_d)^{\beta}) \tag{3.10}$$

This condition insures that all moments of the energy spectrum are finite:

$$\int_0^\infty dk\ k^n E(k) < \infty \tag{3.11}$$

Eq. (3.11) is equivalent to the finiteness of all derivative moments,

$$\langle(\partial^n u/\partial r^n)^2\rangle$$

as discussed by Batchelor [2]. Of the various constants in Eq. (3.10), only the value $\beta = 1$ is widely accepted (note that Orszag [55] has shown that necessarily $\beta \leq 1$).

3.3 Possibility of an equipartition spectrum

Analytically, this is simply the requirement that even if the energy flux vanishes identically, the *modal energy density* defined in equation (2.29) in Section 2 takes the *equipartition* form $\mathsf{U}(\mathbf{k}) = \text{constant}$; equivalently, the energy spectrum is of the form

$$E(k) \sim k^2 \tag{3.12}$$

Note that this equipartition state is not a limit of the Kolmogorov steady state as the dissipation rate approaches zero, for in that limit simply $E(k) \equiv 0$.

The question again arises how such a state can be developed and maintained. The equipartition spectrum is a solution of a modified Lin equation, often called the *inviscid truncated system*, or *truncated Euler equations*, in which viscosity is removed and a high wavenumber cutoff is imposed. If dissipation vanishes, a steady state requires zero production; hence. we have the unsteady equation

$$\dot{E}(k) = -\frac{\partial}{\partial k}\mathcal{F}(k), \quad k_0 \leq k \leq k_1 \tag{3.13}$$

with the zero flux boundary condition

$$\mathcal{F}(k_0) = \mathcal{F}(k_1) = 0 \tag{3.14}$$

This system must evolve to the steady state Eq. (3.12). The restriction to $k_1 < \infty$ in Eq. (3.13) is indispensable; without the imposed maximum wavenumber k_1, this system would exhibit the 'ultraviolet catastrophe' of classical thermodynamics (Gamow [27]; the ultraviolet catastrophe is convincingly illustrated in the figure on p. 16 of Gamow's textbook). Thus, if we have equipartition on an interval, say $0 \leq k \leq k_1$, and then replace k_1 by $k_2 > k_1$, in principle a new equipartition spectrum develops on this larger interval. Energy conservation then implies that $U(k)$ now takes a smaller value, and will become vanishingly small if k_1 is increased indefinitely.

Although both the Kolmogorov and the equipartition spectra are formally constant flux states in which the flux is nonzero and zero respectively, in a Kolmogorov steady state, energy gain from modes with wavenumbers smaller than k is balanced by energy loss to modes with wavenumbers greater than k ($\mathcal{F} > 0$). In the equipartition solution, there is neither net gain nor net loss from either set of modes ($\mathcal{F} = 0$), a condition which is called *detailed balance.*

We will only verify the possibility of an equipartition spectrum; the condition that the inviscid truncated system evolves toward one from arbitrary initial conditions requires the proof of an *H-theorem.* Although an *H*-theorem was proven for the EDQNM closure by Carnevale [11], these more elementary models do not seem to admit *H*-theorems because of the absence of an entropy function. However, we will leave this as an open problem for hypothetical future research.

The equipartition solution of the inviscid truncated system, Eq. (3.12), is a consequence of equilibrium thermodynamics that must be considered exact. The proof that equipartition is a property of the truncated Euler equations was provided by Lee [44]; the idea has a long history, going at least as far back as Onsager [23]. It has important applications [40] to 2D turbulence, helicity conservation, and magnetohydrodynamics among other topics. Nevertheless, these facts alone would not justify imposing the equipartition solution as a model constraint, because the inviscid truncated system is highly idealized. Rather, consistency with equipartition proves to have a much wider significance than its apparently narrow and technical formulation suggests.

Thus, in Kolmogorov's theory, energy is transferred from large scales to small scales: as noted earlier, with the sign conventions of Eq. (3.1), it means that the energy flux is *positive* in a Kolmogorov steady state. But in transient conditions, energy is not necessarily always transferred from large to small scales: for example, an energy spectrum which is initially more shallow than a Kolmogorov spectrum might be expected to relax to a Kolmogorov spectrum in part by transferring energy from small to large scales. Analytically, this will require negative values of the energy flux.

It is evident that establishment of energy equipartition from an arbitrary initial condition, even in the idealized inviscid truncated system, is consistent with an arbitrary sign of $\mathcal{F}$. Then imposing the equipartition constraint is