Jordan Canonical Form and Dynamic Systems on Time Scales

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By

Svetlin G. Georgiev

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Preface

The Jordan canonical form is one of the most important and useful concepts in linear algebra. This book is a development of the Jordan canonical form. The book is intended for senior undergraduate students and beginning graduate students of engineering and science courses. There are seven chapters in this book. The chapters in the book are pedagogically organized. Each chapter concludes with a section of practical problems.

In Chapter 1 we present necessary background material for vector spaces. Some of their properties are deducted. We introduce vector subspaces, sums of spaces and direct sums of spaces. Chapter 2 deals with finite-dimensional vector spaces. In Chapter 3 we introduce linear maps and we deduct some algebraic properties of the linear space $\mathscr{L}(V,W)$ for arbitrary vector spaces V and W. We define matrices and represent a linear map by a matrix. Chapter 4 is devoted to invertible linear maps and isomorphic vector spaces. They are defined products and quotients of vector spaces. Invariant subspaces, eigenvalues and eigenvectors are introduced in Chapter 5. They are investigated restrictions, quotient operators and upper-triangular matrices. In Chapter 6 we define kernels of powers of an operator, nilpotent operators, decomposition of operators, block diagonal matrices and we prove the Cayley-Hamilton theorem. We define characteristic and minimal polynomials for a linear operator. We present an algorithm to find the Jordan canonical form of a matrix. In Chapter 7 we show how to use the Jordan canonical form to solve first order linear dynamic systems on arbitrary times scales.

This book is addressed to a wide audience of specialists such as mathematicians, physicists, engineers and biologists. It can be used as a textbook at the graduate level and as a reference book for several disciplines.

Paris, November 2022 *Svetlin G. Georgiev*

Chapter 1

Vector Spaces

Throughout this book $\mathbb{F}^n$ stands for either $\mathbb{R}^n$ or $\mathbb{C}^n$. The elements of $\mathbb{F}^n$ are called scalars.

1.1 Definition of a Vector Space

Definition 1.1. *An addition on a set V is a function that assigns an element $u+v \in V$ for each pair of elements $u, v \in V$.*

Definition 1.2. *A scalar multiplication on a set V is a function that assigns an element $\lambda v \in V$ for each $\lambda \in \mathbb{F}$ and each $v \in V$.*

Now, we are ready to give a formal definition of a vector space.

Definition 1.3. *A vector space is a set V along an addition on V and a scalar multiplication on V such that the following properties hold:*

1. *(commutativity)*
$$u+v = v+u, \quad u, v \in V.$$

2. *(associativity)*
$$u+(v+w) = (u+v)+w, \quad u, v, w \in V.$$

3. *(additive identity)*
there exists an element $0 \in V$ such that
$$v+0 = v, \quad v \in V.$$

4. *(additive inverse)*

   for any $v \in V$, there exists $w \in V$ such that

$$v + w = 0.$$

5. *(multiplicative identity)*

$$1v = v, \quad v \in V.$$

6. *(distributive properties)*

$$\begin{aligned} a(u+v) &= au+av, \\ (a+b)u &= au+bu \end{aligned}$$

   for any $a, b \in \mathbb{F}$ and any $u, v \in V$.

The elements of a vector space are called vectors or points.

The scalar multiplication in a vector space depends on $\mathbb{F}$. Thus, when we need to be precise, we will say that V is a vector space over $\mathbb{F}$ instead of saying simply V is a vector space. For example, $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, $\mathbb{C}^n$ is a vector space over $\mathbb{C}$.

Definition 1.4. *A vector space over $\mathbb{R}$ is called a real vector space.*

Definition 1.5. *A vector space over $\mathbb{C}$ is said to be a complex vector space.*

The simplest vector space contains only one point. In other words, $\{0\}$ is a vector space. With the usual operations of addition or scalar multiplication, $\mathbb{F}^n$ is a vector space. The example of $\mathbb{F}^n$ motivated our definition of a vector space.

Example 1.6. *F^∞ is defined to be the set of all sequences of elements of $\mathbb{F}$:*

$$\mathbb{F}^\infty = \{(x_1, x_2, \ldots) : x_j \in \mathbb{F}, \quad j \in \{1, 2, \ldots\}\}.$$

Addition and scalar multiplication on $\mathbb{F}^\infty$ are defined as follows:

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) \quad = \quad (x_1 + y_1, x_2 + y_2, \ldots),$$

$$\lambda(x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \ldots).$$

With these definitions we will show that $\mathbb{F}^\infty$ *becomes a vector space over* $\mathbb{F}$. *Suppose that*

$$x = (x_1, x_2, \ldots), \quad y = (y_1, y_2, \ldots), \quad z = (z_1, z_2, \ldots) \in \mathbb{F}^\infty$$

and $a, b \in \mathbb{F}$. *Then, we have the following.*

1.

$$\begin{aligned} x + y &= (x_1, x_2, \ldots) + (y_1, y_2, \ldots) \\ &= (x_1 + y_1, x_2 + y_2, \ldots) \\ &= (y_1 + x_1, y_2 + x_2, \ldots) \\ &= (y_1, y_2, \ldots) + (x_1, x_2, \ldots) \\ &= y + x. \end{aligned}$$

2.

$$\begin{aligned} x + (y + z) &= (x_1, x_2, \ldots) + ((y_1, y_2, \ldots) + (z_1, z_2, \ldots)) \\ &= (x_1, x_2, \ldots) + (y_1 + z_1, y_2 + z_2, \ldots) \\ &= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \ldots) \\ &= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, \ldots) \\ &= (x_1 + y_1, x_2 + y_2, \ldots) + (z_1, z_2, \ldots) \\ &= ((x_1, x_2, \ldots) + (y_1, y_2, \ldots)) + (z_1, z_2, \ldots) \\ &= (x + y) + z. \end{aligned}$$

3. *Let*

$$0 = (0, 0, \ldots).$$

Then

$$\begin{aligned} x + 0 &= (x_1, x_2, \ldots) + (0, 0, \ldots) \\ &= (x_1 + 0, x_2 + 0, \ldots) \\ &= (x_1, x_2, \ldots) \\ &= x. \end{aligned}$$

4. *Let*

$$w = (-x_1, -x_2, \ldots).$$

Then

$$\begin{aligned} x + w &= (x_1, x_2, \ldots) + (-x_1, -x_2, \ldots) \\ &= (x_1 + (-x_1), x_2 + (-x_2), \ldots) \\ &= (0, 0, \ldots). \end{aligned}$$

5.

$$\begin{aligned} 1x &= 1(x_1, x_2, \ldots) \\ &= (1x_1, 1x_2, \ldots) \\ &= (x_1, x_2, \ldots) \\ &= x. \end{aligned}$$

6.

$$a(x + y) = a((x_1, x_2, \ldots) + (y_1, y_2, \ldots))$$

$$
\begin{aligned}
&= a(x_1+y_1, x_2+y_2, \ldots) \\
&= (a(x_1+y_1), a(x_2+y_2), \ldots) \\
&= (ax_1+ay_1, ax_2+ay_2, \ldots) \\
&= (ax_1, ax_2, \ldots) + (ay_1, ay_2, \ldots) \\
&= a(x_1, x_2, \ldots) + a(y_1, y_2, \ldots) \\
&= ax + ay.
\end{aligned}
$$

Next,

$$
\begin{aligned}
a(x+y) &= a((x_1, x_2, \ldots) + (y_1, y_2, \ldots)) \\
&= a(x_1+y_1, x_2+y_2, \ldots) \\
&= (a(x_1+y_1), a(x_2+y_2), \ldots) \\
&= (ax_1+ay_1, ax_2+ay_2, \ldots) \\
&= (ax_1, ax_2, \ldots) + (ay_1, ay_2, \ldots) \\
&= a(x_1, x_2, \ldots) + a(y_1, y_2, \ldots) \\
&= ax + ay.
\end{aligned}
$$

Definition 1.7. *Let S be a set. With $\mathbb{F}^S$ we will denote the set of functions from S to $\mathbb{F}$. For $f, g \in \mathbb{F}^S$, the sum $f+g \in \mathbb{F}^S$ is the function defined by*

$$(f+g)(x) = f(x) + g(x), \quad x \in S.$$

For $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the product $\lambda f \in \mathbb{F}^S$ is the function

$$(\lambda f)(x) = \lambda f(x), \quad x \in S.$$

Example 1.8. $\mathbb{R}^{[0,1]}$ *is the set of all functions from* $[0,1]$ *to* $\mathbb{R}$.

Example 1.9. $\mathbb{F}^S$ *is a vector space over* $\mathbb{F}$. *Let* $f,g,h \in \mathbb{F}^S$ *and* $a,b \in \mathbb{F}$. *Then we have the following.*

1.

$$\begin{aligned}(f+g)(x) &= f(x)+g(x) \\ &= g(x)+f(x) \\ &= (g+f)(x), \quad x \in S.\end{aligned}$$

2.

$$\begin{aligned}(f+(g+h))(x) &= f(x)+(g+h)(x) \\ &= f(x)+(g(x)+h(x)) \\ &= (f(x)+g(x))+h(x) \\ &= (f+g)(x)+h(x) \\ &= ((f+g)+h)(x), \quad x \in S.\end{aligned}$$

3. *Define* $0 : S \to \mathbb{F}$ *as follows:*

$$0(x) = 0, \quad x \in S.$$

Then

$$\begin{aligned}(f+0)(x) &= f(x)+0(x) \\ &= f(x)+0 \\ &= f(x), \quad x \in S.\end{aligned}$$

4. *Define* $-f : S \to \mathbb{F}$ *as follows:*

$$(-f)(x) = -f(x), \quad x \in S.$$

Then

$$\begin{aligned}
(f+(-f))(x) &= f(x)+(-f)(x) \\
&= f(x)-f(x) \\
&= 0 \\
&= 0(x), \quad x \in S.
\end{aligned}$$

5.

$$\begin{aligned}
(1f)(x) &= 1f(x) \\
&= f(x), \quad x \in S.
\end{aligned}$$

6.

$$\begin{aligned}
(a(f+g))(x) &= a(f+g)(x) \\
&= a(f(x)+g(x)) \\
&= af(x)+ag(x) \\
&= (af)(x)+(ag)(x) \\
&= (af+ag)(x), \quad x \in S.
\end{aligned}$$

Next,

$$\begin{aligned}
((a+b)f)(x) &= (a+b)f(x) \\
&= af(x)+bf(x)
\end{aligned}$$

$$= (af)(x)+(bf)(x)$$

$$= (af+bf)(x), \quad x \in S.$$

In fact, the elements of $\mathbb{R}^{[0,1]}$ are real-valued functions on $[0,1]$, not lists. In general, a vector space is an abstract entity whose elements might be lists, functions or weird objects.

Our previous examples of vector spaces, $\mathbb{F}^n$ and $\mathbb{F}^\infty$, are special cases of the vector space $\mathbb{F}^S$ because a list of length n of numbers in $\mathbb{F}$ can be thought of as a function from $\{1,2,\ldots,n\}$ to $\mathbb{F}$ and a sequence of numbers in $\mathbb{F}$ can be thought of as a function from the set of positive integers to $\mathbb{F}$. In other words, we can think of $\mathbb{F}^n$ as $\mathbb{F}^{\{1,2,\ldots,n\}}$ and we can think of $\mathbb{F}^\infty$ as $\mathbb{F}^{\{1,2,\ldots\}}$.

1.2 Elementary Properties of Vector Spaces

In this section, we will deduce some properties of vector spaces.

Theorem 1.10. *A vector space has a unique additive identity.*

Proof. Suppose that 0_1 and 0_2 are two additive identities in the vector space V. Then

$$\begin{aligned} 0_1 &= 0_1+0_2 \\ &= 0_2+0_1 \\ &= 0_2. \end{aligned}$$

This completes the proof. □

Theorem 1.11. *Every element in a vector space has a unique additive inverse.*

Proof. Suppose that V is a vector space. Let $v \in V$ and $w_1, w_2 \in V$ be its additive inverse. Then

$$v+w_1 = 0,$$

$$v + w_2 = 0$$

and

$$\begin{aligned} w_1 &= w_1 + 0 \\ &= w_1 + (v + w_2) \\ &= (w_1 + v) + w_2 \\ &= (v + w_1) + w_2 \\ &= 0 + w_2 \\ &= w_2 + 0 \\ &= w_2. \end{aligned}$$

This completes the proof. □

Below, suppose that V is a vector space. Because additive inverses are unique, the following notation makes sense. Let $v, w \in V$.

1. $-v$ denotes the additive inverse of v.
2. $w - v$ is defined to be $w + (-v)$.

Theorem 1.12. *We have*

$$0v = 0, \quad v \in V,$$

where 0 *denotes a scalar (the number* $0 \in \mathbb{F}$*) on the left side of the equation and a vector(the additive identity of* V*) on right side of the equation.*

Proof. For $v \in V$, we have

$$\begin{aligned} 0v &= (0 + 0)v \\ &= 0v + 0v, \end{aligned}$$

whereupon

$$0v = 0.$$

This completes the proof. □

Theorem 1.13. *We have*

$$a0 = 0, \quad a \in \mathbb{F}, \tag{1.1}$$

where $0 \in V$ *is the additive identity of* V.

Proof. For $a \in \mathbb{F}$, we have

$$\begin{aligned} a0 &= a(0+0) \\ &= a0 + a0, \end{aligned}$$

whereupon we get (1.1). This completes the proof. □

Exercise 1.14. *Suppose that* $a \in \mathbb{F}$*,* $v \in V$*, and* $av = 0$*. Prove that* $a = 0$ *or* $v = 0$*.*

Theorem 1.15. *We have*

$$(-1)v = -v, \quad v \in V.$$

Proof. For $v \in V$, we have

$$\begin{aligned} v + (-1)v &= 1v + (-1)v \\ &= (1 + (-1))v \\ &= 0v \\ &= 0. \end{aligned}$$

Thus, $(-1)v$ is the additive inverse of v. This completes the proof. □

Exercise 1.16. *Prove that*

$$-(-v) = v, \quad v \in V.$$

Exercise 1.17. *Suppose that* $v, w \in V$*. Explain why there exists a unique* $x \in V$ *such that*

$$v + 3x = w.$$

1.3 Subspaces

Definition 1.18. *A subset U of V is called a subspace of V if U is also a vector space using the same addition and scalar multiplication as on V.*

Some mathematicians, use the term ”linear subspace”, which means the same as subspace.

Example 1.19. *The set*

$$\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{F}\}$$

is a subspace of $\mathbb{F}^3$.

Theorem 1.20. *A subset U of V is a subspace if and only if U satisfies the following three conditions:*

1. $0 \in U$.
2. *if $u, v \in U$, then $u + v \in U$.*
3. *if $a \in \mathbb{F}$ and $u \in U$, then $au \in U$.*

Proof. 1. Let U be a subspace of V. Then, by the definition for a subspace and by the definition for a vector space, it follows that U satisfies 1, 2, 3.

2. Let U satisfies 1, 2, 3. Take $u, v, w \in U$ and $a, b \in \mathbb{F}$ arbitrarily. Then $u, v, w \in V$ and hence,

$$\begin{aligned} u + v &= v + u, \\ u + (v + w) &= (u + v) + w, \\ a(u + v) &= au + av, \\ (a + b)u &= au + bu, \\ u + 0 &= u. \end{aligned}$$

By the third condition, it follows that $(-1)u \in U$ and

$$u - u = u + (-1)u$$

$$= \quad 0.$$

Thus, U is a subspace of V. This completes the proof.

□

Remark 1.21. *The additive identity condition above could be replaced with the condition that U is nonempty. Then, taking $u \in U$, multiplying it by 0, and using the condition that U is closed under scalar multiplication would imply that $0 \in U$. However, if U is indeed a subspace of V, then the easiest way to show that U is nonempty is to show that $0 \in U$.*

Remark 1.22. *The three conditions in the result above usually enable us to determine quickly whether a given subset of V is a subspace of V.*

Example 1.23. *Consider the set*

$$U = \{(u_1, u_2, u_3, u_4) \in \mathbb{F}^4 : u_3 = 5u_4\}.$$

Let

$$u = (u_1, u_2, u_3, u_4), \quad v = (v_1, v_2, v_3, v_4) \in \mathbb{F}^4$$

and $a \in \mathbb{F}$ be arbitrarily chosen. Then

$$\begin{aligned} u_3 &= 5u_4, \\ v_3 &= 5v_4, \\ au_3 &= 5au_4 \end{aligned}$$

and

$$u_3 + v_3 = 5(u_4 + v_4).$$

Therefore $0 \in U$, $u + v \in U$ and $au \in U$. Hence and Theorem 1.20, it follows that U is a subspace of $\mathbb{F}^4$.

Note that $\{0\}$ is the smallest subspace of V and V is the largest subspace of V. Moreover, the empty set is not a subspace because it must contain an additive identity.

Example 1.24. *Let U be the set of real-valued continuous functions on $[0, 1]$. Let also, $u, v \in U$ and $a \in \mathbb{F}$. Then, we have the following.*

1. *Since 0 is a continuous function on $[0, 1]$, then $0 \in U$.*

2. *Since the sum of two continuous functions u and v on $[0,1]$ is a continuous function on $[0,1]$, we have that $u+v \in U$.*

3. *Since the multiplication of a continuous function $u \in U$ on $[0,1]$ by a scalar $a \in \mathbb{F}$ is a continuous function on $[0,1]$, we get $au \in \mathbb{F}$.*

Now, applying Theorem 1.20, we get that U is a subspace of $\mathbb{R}^{[0,1]}$.

Exercise 1.25. *Prove that the set of all differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.*

1.4 Sum of Spaces

Definition 1.26. *Suppose that U_1, ..., U_m are subsets of V. The sum of U_1, ..., U_m, denoted by*

$$U_1 + \cdots + U_m,$$

is the set of all possible sums of elements of U_1, ..., U_m. More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}.$$

Example 1.27. *Let*

$$U = \{(x,0,0) \in \mathbb{F}^3 : x \in \mathbb{F}\},$$

$$W = \{(0,y,0) \in \mathbb{F}^3 : y \in \mathbb{F}\}.$$

Take

$$(x,0,0) \in U, \quad (0,y,0) \in W.$$

Then

$$(x,0,0) + (0,y,0) = (x,y,0).$$

Therefore

$$U + W = \{(x,y,0) \in \mathbb{F}^3 : x,y \in \mathbb{F}\}.$$

Example 1.28. *Let*

$$U = \{(x,x,y,y) \in \mathbb{F}^4 : x,y \in \mathbb{F}\},$$

$$W = \{(x,x,x,y) \in \mathbb{F}^4 : x,y \in \mathbb{F}\}.$$

Take

$$(x_1,x_1,y_1,y_1) \in U, \quad (x_2,x_2,x_2,y_2) \in W.$$

Then

$$(x_1,x_1,y_1,y_1)+(x_2,x_2,x_2,y_2) = (x_1+x_2,x_1+x_2,y_1+x_2,y_1+y_2).$$

Therefore

$$U+W = \{(x,x,y,z) \in \mathbb{F}^4 : x,y,z \in \mathbb{F}\}.$$

Exercise 1.29. *Let*

$$U = \{(x,x,x,x,x) \in \mathbb{F}^5 : x \in \mathbb{F}\},$$

$$W = \{(x,y,y,y,x) \in \mathbb{F}^5 : x,y \in \mathbb{F}\}.$$

Find $U+W$.

Theorem 1.30. *Suppose* U_1, ..., U_m *are subspaces of* V. *Then* $U_1+\cdots+U_m$ *is the smallest subspace of* V *containing* U_1, ..., U_m.

Proof. Since $U_1, \ldots, U_m$ are subspaces of V, we have that $0 \in U_1, \ldots, 0 \in U_m$ and then

$$\begin{aligned} 0+\cdots+0 &= 0 \\ &\in U_1+\cdots+U_m. \end{aligned}$$

Let $u, v \in U_1+\cdots+U_m$ be arbitrarily chosen. Then, there are $u_1, v_1 \in U_1$, ..., $u_m, v_m \in U_m$ so that

$$\begin{aligned} u &= u_1+\cdots+u_m, \\ v &= v_1+\cdots+v_m. \end{aligned}$$

Hence,

$$u+v = (u_1+v_1)+\cdots+(u_m+v_m). \tag{1.2}$$

Since U_1 is a subspace of V and $u_1, v_1 \in U_1$, we conclude that $u_1+v_1 \in U_1$, and so on. Because U_m is a subspace of V and $u_m, v_m \in U_m$, then $u_m+v_m \in U_m$. From here and from (1.2), we obtain that $u+v \in U_1+\cdots+U_m$ and

$U_1 + \cdots + U_m$ is closed under addition. Let $a \in F$ be arbitrarily chosen. Then

$$au = au_1 + \cdots + au_m. \tag{1.3}$$

Since U_1 is a subspace of V and $a \in \mathbb{F}$, $u_1 \in U_1$, we obtain $au_1 \in U_1$, and so forth. Because U_m is a subspace of V and $a \in \mathbb{F}$, $u_m \in U_m$, we find $au_m \in U_m$. Hence and (1.3), we conclude that $au \in U_1 + \cdots + U_m$ and $U_1 + \cdots + U_m$ is closed under scalar multiplication. Now, applying Theorem 1.20, we find that $U_1 + \cdots + U_m$ is a subspace of V. Let now, $w_1 \in U_1$ be arbitrarily chosen and fixed. Then, using that $0 \in U_2$, ..., $0 \in U_m$, we find

$$\begin{aligned} w_1 &= w_1 + 0 + \cdots + 0 \\ &\in U_1 + \cdots + U_m. \end{aligned}$$

Because $w_1 \in U_1$ was arbitrarily chosen and we get that it is an element of $U_1 + \cdots + U_m$, we obtain the inclusion

$$U_1 \subseteq U_1 + \cdots + U_m.$$

Let $w_m \in U_m$ be arbitrarily chosen and fixed. Then, using that $0 \in U_1$, ..., $0 \in U_{m-1}$, we obtain

$$\begin{aligned} w_m &= 0 + \cdots + 0 + w_m \\ &\in U_1 + \cdots + U_{m-1} + U_m. \end{aligned}$$

Because $w_m \in U_m$ was arbitrarily chosen and we get that it is an element of $U_1 + \cdots + U_m$, we find the inclusion

$$U_m \subseteq U_1 + \cdots + U_m.$$

Now, let W be a subspace of V so that

$$U_1, \ldots, U_m \subseteq W.$$

Because subspaces must contain all finite sums of their elements, we get

$$U_1 + \cdots + U_m \subseteq W.$$

Thus, $U_1 + \cdots + U_m$ is the smallest subspace of V containing U_1, ..., U_m. This completes the proof. □

Remark 1.31. *Sums of subspaces in the theory of vector spaces are analogous to unions of subsets in the set theory. Given two subspaces of a vector space, the smallest subspace containing them is their sum. Analogously, given two subsets of a set, the smallest subset containing them is their union.*

1.5 Direct Sums

Definition 1.32. *Let $U_1, \ldots, U_m$ be subspaces of V. Then $U_1 + \cdots + U_m$ is called a direct sum if each element of $U_1 + \cdots + U_m$ can be represented in a unique way in the form*

$$u_1 + \cdots + u_m,$$

where $u_1 \in U_1, \ldots, u_m \in U_m$. If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with $\oplus$ notation serving as an indication that this is a direct sum.

Example 1.33. *Let*

$$\begin{aligned} U &= \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}, \\ W &= \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\}. \end{aligned}$$

Note that U and W are subspaces of $\mathbb{F}^3$. Next,

$$U + W = \{(x, y, z) \in \mathbb{F}^3 : x, y, z \in \mathbb{F}\}.$$

Assume that an element $u \in U + W$ has two representations

$$\begin{aligned} u &= (x_1, y_1, 0) + (0, 0, z_1) \in U + W, \\ u &= (x_2, y_2, 0) + (0, 0, z_2) \in U + W, \quad x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{F}. \end{aligned}$$

Then

$$\begin{aligned} (x_1, y_1, 0) + (0, 0, z_1) &= (x_1, y_1, z_1) \\ &= (x_2, y_2, 0) + (0, 0, z_2) \\ &= (x_2, y_2, z_2), \end{aligned}$$

whereupon

$$\begin{aligned} x_1 &= x_2, \\ y_1 &= y_2, \end{aligned}$$

$$z_1 = z_2,$$

and

$$(x_1, y_1, 0) = (x_2, y_2, 0),$$

$$(0, 0, z_1) = (0, 0, z_2).$$

Therefore $U \oplus W = \mathbb{F}^3$.

Example 1.34. *Let*

$$U_1 = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\},$$

$$U_2 = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\},$$

$$U_3 = \{(0, y, y) \in \mathbb{F}^3 : y \in \mathbb{F}\}.$$

We have that U_1, U_2 *and* U_3 *are subspaces of* $\mathbb{F}^3$. *Let*

$$u_1 = (x_1, y_1, 0) \in U_1,$$

$$u_2 = (0, 0, z_2) \in U_2,$$

$$u_3 = (0, y_3, y_3) \in U_3.$$

Then

$$u_1 + u_2 + u_3 = (x_1, y_1, 0) + (0, 0, z_2) + (0, y_3, y_3)$$

$$= (x_1, y_1 + y_3, z_2 + y_3)$$

and

$$U_1 + U_2 + U_3 = \mathbb{F}^3.$$

Note that

$$(0, 0, 0) \in U_1 + U_2 + U_3$$

and

$$(0, 0, 0) \in U_1,$$

$$\begin{aligned}(0,0,0) &\in U_2,\\ (0,0,0) &\in U_3,\\ (0,1,0) &\in U_1,\\ (0,0,1) &\in U_2,\\ (0,-1,-1) &\in U_3.\end{aligned}$$

Moreover,

$$(0,0,0) = (0,0,0) + (0,0,0) + (0,0,0)$$

and

$$(0,0,0) = (0,1,0) + (0,0,1) + (0,-1,-1).$$

Thus, $(0,0,0) \in U_1 + U_2 + U_3$ *has two different representations as a sum* $u_1 + u_2 + u_3$ *with* $u_1 \in U_1$*,* $u_2 \in U_2$ *and* $u_3 \in U_3$*. Therefore* $U_1 + U_2 + U_3$ *is not a direct sum.*

Exercise 1.35. *Let*

$$U = \{(x,y,y) \in \mathbb{F}^3 : x,y \in \mathbb{F}\}.$$

Find a subspace $W \in \mathbb{F}^3$ *so that*

$$U \oplus W = \mathbb{F}^3.$$

Theorem 1.36. *Suppose that* $U_1, \ldots, U_m$ *are subspaces of* V*. Then* $U_1 + \cdots + U_m$ *is a direct sum if and only if the only way to write* 0 *as a sum* $u_1 + \cdots + u_m$ *, where* $u_1 \in U_1, \ldots, u_m \in U_m$*, is by taking each* $u_1, u_2, \ldots, u_m$ *equal to* 0*.*

Proof. 1. Suppose that $U_1 + \cdots + U_m$ is a direct sum. Since $U_1, \ldots, U_m$ are subspaces of V, we have that

$$0 \in U_1, \quad \ldots, \quad 0 \in U_m, \quad 0 \in U_1 + \cdots + U_m.$$

By the definition of a direct sum, it follows that the only way to write 0 as a sum $u_1 + \cdots + u_m$, where $u_1 \in U_1, \ldots, u_m \in U_m$, is by taking each $u_1, u_2, \ldots, u_m$ equal to 0.

2. Suppose that the only way to write 0 as a sum $u_1 + \cdots + u_m$, where $u_1 \in U_1$, ..., $u_m \in U_m$, is by taking each u_1, u_2, ..., u_m equal to 0. Let $v \in U_1 + \cdots + U_m$ be arbitrarily chosen and fixed. Let also,

$$u = u_1 + \cdots + u_m$$

and

$$u = v_1 + \cdots + v_m,$$

with $u_1, v_1 \in U_1$, ..., $u_m, v_m \in U_m$. Subtracting these two equations, we find

$$0 = (u_1 - v_1) + \cdots + (u_m - v_m).$$

Since U_1, ..., U_m are subspaces of V, we have

$$u_1 - v_1 \in U_1, \quad \ldots, \quad u_m - v_m \in U_m.$$

The equation above implies that

$$\begin{aligned} u_1 - v_1 &= 0, \\ &\vdots \\ u_m - v_m &= 0, \end{aligned}$$

whereupon

$$\begin{aligned} u_1 &= v_1, \\ &\vdots \\ u_m &= v_m. \end{aligned}$$

This completes the proof.

□

Theorem 1.37. *Suppose that U_1 and U_2 are subspaces of V. Then $U_1 + U_2$ is a direct sum if and only if $U_1 \cap U_2 = \{0\}$.*

Proof. 1. Suppose that U_1+U_2 is a direct sum. Take $v \in U_1 \cap U_2$ arbitrarily. Then

$$0 = v + (-v),$$

where $v \in U_1$ and $-v \in U_2$. Because U_1+U_2 is a direct sum, we have that 0 has a unique representation. Therefore $v = 0$. Since $v \in U_1 \cap U_2$ was arbitrarily chosen, we conclude that $U_1 \cap U_2 = \{0\}$.

2. Let $U_1 \cap U_2 = \{0\}$. We will prove that U_1+U_2 is a direct sum. Suppose that

$$0 = u_1 + u_2,$$

where $u_1 \in U_1$, $u_2 \in U_2$. Then

$$u_2 = -u_1.$$

Since U_1 is a subspace of V, we have that $-u_1 \in U_1$. Therefore $u_2 \in U_1$ and $u_2 \in U_1 \cap U_2$. From here, $u_2 = 0$ and $u_1 = 0$. Thus, 0 has a unique representation as a sum u_1+u_2 with $u_1 \in U_1$ and $u_2 \in U_2$. Hence and Theorem 1.36, we conclude that U_1+U_2 is a direct sum. This completes the proof.

□

Remark 1.38. *The result above only deals with the case of two subspaces. When asking about a possible direct sum with more than two subspaces, it is not enough to test that each pair of the subspaces intersect only at* $\{0\}$*. To see this, consider Example 1.34. We have that*

$$U_1 \cap U_2 = \{0\}, \quad U_1 \cap U_3 = \{0\}, \quad U_2 \cap U_3 = \{0\}.$$

At the same time, we have that $U_1+U_2+U_3$ *is not a direct sum.*

Remark 1.39. *Sums of subspaces are analogous to unions of subsets. Similarly, direct sums of subspaces are analogous to disjoint unions of subsets. No two subspaces of a vector space can be disjointed, because both contain* 0*. So, disjointness is replaced, at least in the case of two subspaces, with the requirement that the intersection equals* $\{0\}$*.*

1.6 Advanced Practical Problems

Problem 1.40. *Prove that*

$$-(-(-v)) \quad = \quad -v,$$

$$-(-(-(-v))) \quad = \quad v, \quad v \in V.$$

Problem 1.41. *Suppose that $v, w \in V$. Explain why there exists a unique $x \in V$ such that*

$$2v - 5x = 3w.$$

Problem 1.42. *Prove that the set of all differentiable real-valued functions f on the interval $(0,3)$ such that*

$$f'(2) = b$$

is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.

Problem 1.43. *Let ∞ and $-\infty$ be two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define*

$$
\begin{aligned}
t\infty &= \begin{cases} -\infty & \text{if} \quad t < 0 \\ 0 & \text{if} \quad t = 0 \\ \infty & \text{if} \quad t > 0, \end{cases} \\
t(-\infty) &= \begin{cases} \infty & \text{if} \quad t < 0 \\ 0 & \text{if} \quad t = 0 \\ -\infty & \text{if} \quad t > 0, \end{cases} \\
t + \infty &= \infty, \\
\infty + t &= \infty, \\
t + (-\infty) &= -\infty, \\
(-\infty) + t &= -\infty, \\
\infty + \infty &= \infty,
\end{aligned}
$$

$$(-\infty)+(-\infty) = -\infty,$$

$$\infty+(-\infty) = 0.$$

Is $\mathbb{R}\cup\{\infty\}\cup\{-\infty\}$ *a vector space over* $\mathbb{R}$*? Explain.*

Problem 1.44. *Prove that the set*

$$\{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{F}^5 : x_1+x_2=3x_5, \quad x_2=2x_4\}$$

is a subspace of $\mathbb{F}^5$.

Problem 1.45. *Prove that the set*

$$\{(x_1,x_2,x_3) : x_1=x_2+3\}$$

is not a subspace of $\mathbb{F}^3$.

Problem 1.46. *Let*

$$U = \{(x,y,-x)\in\mathbb{F}^3 : x,y\in\mathbb{F}\},$$

$$W = \{(x,2x,-x)\in\mathbb{F}^3 : x\in\mathbb{F}\}.$$

Find $U+W$.

Problem 1.47. *Let*

$$U_1 = \{(x,0,\ldots,0)\in\mathbb{F}^n : x\in\mathbb{F}\},$$

$$U_2 = \{(0,x,\ldots,0)\in\mathbb{F}^n : x\in\mathbb{F}\},$$

$$\vdots$$

$$U_n = \{(0,0,\ldots,0,x)\in\mathbb{F}^n : x\in\mathbb{F}\}.$$

Prove that

$$U_1\oplus\cdots\oplus U_n=\mathbb{F}^n.$$

Problem 1.48. *For each of the following subsets of* $\mathbb{F}^4$*, determine whether it is a subspace of* $\mathbb{F}^4$.